Topology Revised Midterm

1. Give a definition of what topological space is.

An abstract set X is called a topological space (X, τ) if there exists a topology τ defined on X, i.e. there exists a set τ of subsets of X, or in other words, $\tau \subseteq 2^X$, where 2^X is a powerset of X, and τ satisfies the following statements:

1. $\emptyset, X \in \tau$
2. If $\forall \alpha \in A \quad X_\alpha \in \tau$, then $\bigcup\limits_{\alpha \in A} X_\alpha \in \tau$
3. If $X_k \in \tau \ \ \forall k = \overline{1, n}$, then $\bigcap\limits_{k=1}^{n} X_k \in \tau$

2. Let $S \subset \mathbb{R}^n$ be a subset of a topological space $\mathbb{R}^n$, and let S^C denote the complement $\mathbb{R}^n - S$ of S. Show that S is open if and only if every limit point of S^C is contained in S^C.

Let us suppose that S is open. Then by definition $\mathbb{R}^n - S = S^C$ is closed. It is left to show that every limit point of S^C is contained in S^C. Let $S^C = U$ and let the set of limit points be denoted as U'. Then we must show that $U' \subseteq U$. Let $x \in U'$ be an arbitrary point. Then $x \in U$ or $x \in \partial U$. If $x \in U$, then everything is proved. Otherwise, $x \in \partial U$. Let us suppose that $x \notin U$ (otherwise everything is proved, again). As $x \notin U$, then $x \in \mathbb{R}^n - U = S$. S in an open set, so there exists an open ball $B(x) : B(x) \subset S$. From another point of view, $x \in \partial\Omega$ and any open ball with the center x has nonempty intersection with U, so $B(x) \cap U \neq \emptyset$. Let us denote this intersection as V. It's obvious that $V \subset U$ and $V \subset B(x) \subset S$, so $V \subset S$. But U and S are disjoint (by definition), and that's why a proposition that $x \notin U$ was wrong and $x \in U$. Q.E.D.

Conversely, let every limit point of S^C be contained in S^C. Let us prove that S is an open set. To do it, we can prove that S^C is a closed set. We assume contrary, i.e. $S^C = U$ is an open set. If every limit point is contained in U, then $\partial U \subset U$. As U is an open set, then for any $x \in U$ there exists a ball $B(x)$:$B(x) \subset U$. Let us take $x \in \partial U$. By definition, there exists a ball $B(x)$:$B(x) \subset U$. From another point of view, $x \in \partial U$ and any ball $B(x)$ has nonempty intersection with complement of U, which is S. Let $B(x) \cap S = V$. Then $V \subset B(x)$, but $B(x) \subset U$. So, $V \subset U$, but $V \in S$, and since S and U are disjoint, we have come to contradiction, i.e. S^C is a closed set and S is an open set. Q.E.D.

3. Which of the following subsets of $[0; 1) \subset \mathbb{R}$ are open in $[0; 1)$(as a subspace of $\mathbb{R}$)? Explain.

(a) $[0, \frac{1}{2})$
(b) $[\frac{1}{2}, 1)$
(c) $\{\frac{1}{2}\}$
(d) $[0, 1)$
(e) $(0, 1)$

Only (a),(d),(e) satisfy given condition, because for any of the other cases of sets we can choose a point and a ball with the center at this point, so that its intersection with a complement of the given set is nonempty. (b) $x = \frac{1}{2}$. (c) $x = \frac{1}{2}$. For the rest cases we will show why given sets are open.

(a) Let $X = [0, \frac{1}{2})$. Then $[0; 1) - X = [\frac{1}{2}, 1)$. As we can see, for any $x \in [0, \frac{1}{2})$ there exists a ball $B(x)$ which is a proper subset of $[0, \frac{1}{2})$.

(d) Let $X = [0, 1)$. Then this set is open by definition of a topological subspace.

(e) Let $X = (0, 1)$. Then its complement is $\{0\}$ a closed set in a topology of subspace.

4. Give an example of a subset $A \subset \mathbb{R}^2$ such that A is neither open nor closed.

$A = B(0, 0) \cup (1, 0)$, where the radius is equal to 1.

5. Give an example of a subset $A \subset \mathbb{R}^2$ such that A both closed and unbounded.

$A = \mathbb{R} \times [0, +\infty)$

6. Give a definition of what it means for a map $F : X \to Y$ between topological spaces X and Y to be continuous.

A mapping $F : X \to Y$ between topological spaces (X, τ) and (Y, μ) is continuous if any inverse image of $V \in \mu$ is an element U of topology τ, in other words

$$\forall V \in \mu \ \ \exists U \in \tau : F(U) = V$$

7. Give an example of a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ which is continuous.

For instance, $F(x, y) = (x - 1, y + 1)$ a shift. We can check the continuity only for bases of the topologies. Indeed, it's obvious that any ball $B(x, y) \in \mu$ has an inverse image $B(x + 1, y - 1) \in \tau$, which says that this function is continuous for bases of corresponding topologies, and therefore, for spaces as well.

8. Give an example of a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ which is not continuous.

$$F(x, y) = \begin{cases} (\dfrac{1}{x}, y), x \neq 0 \\ (0, y), x = 0 \end{cases}$$

9. What does it mean for a space to have the fixed-point property? (Give the definition).

A topological space (X, τ) has a fixed-point property, if any continuous function $f : X \to X$ has a fixed point.

10. Give an example of a space which does not have the fixed-point property.

Let $(X, \tau) = ((0, 1), \tau)$, where τ is a standard topology on $\mathbb{R}$. Then a mapping $f(x) = x^2$ has no fixed points.

11. What does it mean for a space to be path connected? (Give a definition, including a definition of what a path is.)

A topological space (X, τ) is path connected, if for any points $x, y \in X$ there exists a path with x, y at the beginning and at the end. A path is an image of $[0, 1] \subset \mathbb{R}$ with respect to a continuous function $f : [0, 1] \to X$.

12. Give an example of a space which is path connected.

$\mathbb{R}^2$ with standard topology is path connected, since any two points can be connected with a help of a segment of a line, which is homeomorphic to $[0,1]$

13. Give a general definition of what it means for a space to be compact.

A topological space (X,τ) is called compact, if every open-set cover $X = \bigcup\limits_{\alpha\in A} U_\alpha, U_\alpha \in \tau$ has a finite subcover $X = \bigcup\limits_{k=1}^{n} U_k, U_k \in \tau$.

14. Give an example of an open covering $\mathbb{R}^2 \subset \bigcup\limits_i V_i$ of $\mathbb{R}^2$ such that each of the open sets V_i is bounded.

Let us take, for instance, $\mathbb{R}^2 = \bigcup\limits_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} B(i,j)$, where the radius of each ball is equal to 2.

15. Give an example of a subset $K \subset \mathbb{R}^2$ such that K is closed, but K is not compact.

Since $K \subset \mathbb{R}^2$ is compact if and only if it is closed and bounded, we can take a closed unbounded set, as in (5).

16. What is the definition of homeomorphism?

A mapping $F : X \to Y$ of topological spaces (X,τ) and (Y,μ) is called a homeomorphism, if F is continuous and has an inverse mapping F^{-1}, which is also continuous.

17. Give an example of homeomorphism

We can take, for instance, the same function as in (7). F and F^{-1} are inverse to each other and continuous.

18. Give an example of a bounded subset $B \subset \mathbb{R}$ and an unbounded subset $U \subset \mathbb{R}$ such that B and U are homeomorphic.

Let us take $B = \left(0, \frac{\pi}{2}\right)$ and $U = (0, +\infty)$. Here $f : B \to U, f(x) = \tan x$. This function is continuous and has a continuous inverse $f^{-1} : U \to B,\ f^{-1}(y) = \arctan y$.

19. Give an example of a continuous map which is one-to-one and into, but is not a homeomorphism.

We take $f : X \to X$, in details $f : (X,\tau) \to (X,\mu)$, where τ is a discrete topology on X, and μ is any other topology on X. Direct mapping is continuous, while its inverse is not.

20. Describe five subspaces of $\mathbb{R}^3$ such that each is not homeomorphic to any of the others. Justify your answer.

Here we recall the notion of homotopic equivalence. If topological spaces are homeomorphic, then they are homotopy equivalent (this is a necessary condition). So, if we take 5 spaces which are not homotopy equivalent in pairs, then they are not homeomorphic in pairs. So, we can take a torus, double torus, triple torus and so on up to 5-fold torus. Each of them is not homeomorphic to any of the others, since they have different homotopy types.

Work citation:

Algebraic Topology, given by N J Wildberger at UNSW.

Youtube link: **https://www.youtube.com/watch?v=oYFZaqArf54&index=39&list=PL6763F57A61FE6FE8**